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The influence of simple supports on the radiation from turbulent flow near a plane compliant surface

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The paper deals with an extension of previous work on the radiation properties of turbulent flow formed on compliant surfaces. The effect of simple supports is shown to be acoustically equivalent to an extended dipole system of strength equal to the support stress. The dipole radiation is reduced by a transmission factor below that radiated into a uniform environment. A particular example is worked out in detail. That example deals with the case of a single point support on an otherwise homogeneous surface excited by boundary-layer turbulence.

1. The dipole equivalent of simple supports

The influence of surface vibration on the radiation from turbulent flow near a homogeneous plane surface has recently been treated by Ffowcs Williams (1965). There, it was shown that surface response did not introduce sources of high efficiency, and that any surface effect could be accounted for by a straightforward reflexion coefficient for plane acoustic waves. The influence of simple supports can be treated in a similar way. Again, non-linear terms in surface response are neglected, as are the viscous terms. The equations that describe the radiation field are those given by Powell (1960). There are two complementary equations, one for a real flow with turbulence stress tensor T_{ij} distributed over the volume v_+, and one for a hypothetical image flow with a specular reflexion of the turbulence in the volume v_-.

$$\left.\begin{aligned} p(\mathbf{x},t) &= T_+ + p_r + p_v, \quad 0 = T_- - p_r + p_v,\\ T_\pm &= \frac{1}{4\pi}\frac{\partial^2}{\partial x_i\,\partial x_j}\int_{v_\pm}[T_{ij}]\frac{d\mathbf{y}}{r};\\ p_r &= \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_s [p]\frac{d\mathbf{y}}{r}; \quad p_v = -\frac{1}{4\pi}\int_s \bar{\rho}\frac{\partial[v_n]}{\partial t}\frac{d\mathbf{y}}{r}.\end{aligned}\right\} \tag{1.1}$$

The brackets [ ], indicate that the integrals should be evaluated at retarded time, $(t-r/a_0)$, r being the distance separating the source point $\mathbf{y}$ from the observer at $\mathbf{x}$, and a_0 the speed of sound. n is the outward normal from the real flow through the plane bounding surface s. $p(\mathbf{x},t)$ is the pressure radiated to the point $(\mathbf{x},t)$ and comprises four distinct source terms. The quadrupoles in the real flow induce a pressure T_+ while those in the image flow induce a pressure T_-. Should the surface be rigid, surface sources would account for a pressure p_r, while a pressure p_v would be radiated by surface terms if the surface were perfectly limp and could support no stress. More general surfaces radiate sound in a way that is determined by equation (1.1), so that the problem is reduced to one of relating the two pressures p_r and p_v through some knowledge of the surface response.

We now suppose that the previously considered homogeneous plane surface is supported by an inhomogeneous stress system induced by a distribution of simple supports. These stresses will be denoted by q, a positive q implying a force acting on unit area of the surface in the direction $-n$. The stresses are related to the response velocity v_n, through the response equation

$$p-q = F(v_n), \tag{1.2}$$

F being a collection of differential or integral operators representing a linear integro-differential equation with constant coefficients. The surface pressure is now eliminated from (1.1) by use of the response equation

$$p_r = \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_s [p]\frac{d\mathbf{y}}{r} = \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_s [q]\frac{d\mathbf{y}}{r}+\frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_s [F(v_n)]\frac{d\mathbf{y}}{r}. \tag{1.3}$$

This relation reduces the current problem, by analogy, to that of the unsupported homogeneous surface. This becomes clear when we regard the support stress q as the strength of a distribution of externally applied acoustic dipoles, whose total strength we do not expect to be generally zero. These external dipoles are essentially different from the remaining surface terms, representing real sources of radiation and not, in general, accounting for a reflexion property which is the sole role of the other terms. This point will become clear in what follows, but we anticipate it by combining the total effect of the applied stress fields into one term. That term represents the sum of the pressure induced by the turbulent flow and that induced by the support dipoles, a value we denote by S. As before, we have a field due to the real and image source systems

$$S_+ = T_+ + \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_s [q]\frac{d\mathbf{y}}{r}, \quad S_- = T_- - \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_s [q]\frac{d\mathbf{y}}{r}. \tag{1.4}$$

S_- is the pressure induced by the specular reflexion of the real source system that generates the pressure S_+, the reflexion of the dipole term merely requiring a change of sign.

Equations (1.1), (1.3) and (1.4) can be combined in a form that makes clear the analogy with the earlier problem:

$$\left.\begin{aligned} p(\mathbf{x},t) &= S_+ + \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_s [F(v_n)]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_s \bar{\rho}\frac{\partial}{\partial t}[v_n]\frac{d\mathbf{y}}{r}, \\ 0 &= S_- - \frac{1}{4\pi}\frac{\partial}{\partial x_n}\int_s [F(v_n)]\frac{d\mathbf{y}}{r} - \frac{1}{4\pi}\int_s \bar{\rho}\frac{\partial}{\partial t}[v_n]\frac{d\mathbf{y}}{r}. \end{aligned}\right\} \tag{1.5}$$

This system of equations is precisely that treated by Ffowcs Williams (1965) in considering the problem of turbulent flow formed on an unsupported homogeneous surface. In fact, the only change induced by the supports is that the turbulent sources are reinforced by surface dipoles so that S replaces T. Conclusions can therefore be based directly on that analysis. The most important point is that the surface integrals account for simple reflexion of the source system S_+. However, the reflexion coefficient changes with direction of radiation and frequency. The analysis is particularly simple for the distant radiation field, where $p^*(\mathbf{x}, \omega)$, the component of radiated pressure at frequency ω, is given by the sum of direct and reflected fields:

$$p^*(\mathbf{x}, \omega) = S_+^* + RS_-^*. \tag{1.6}$$

S_+^* and S_-^* are the components of sound pressure radiated by the real and image source systems at frequency ω, and R is the reflexion coefficient for plane acoustic waves at that frequency.

The pressures S_+^* and S_-^* consist of a superposition of the fields induced by quadrupoles acoustically equivalent to the turbulent flow and dipoles whose strength density equals the supporting stresses. Equation (1.6) has an interesting special case when there is negligible turbulence so that both T_+ and T_- are zero. Then S_+^* is entirely due to an externally applied stress and S_-^* is its exact opposite, being the field of an image dipole. The radiated pressure is then given by

$$p^*(\mathbf{x}, \omega) = (1-R)\, S_+^*. \tag{1.7}$$

$(1-R)$ is familiar as the transmission coefficient for waves passing from the fluid into a region with impedance equal to the normal impedance of the surface. This result, that the radiation from an externally excited surface is equal to that induced by dipoles of strength density equal to the applied stresses multiplied by the transmission coefficient for plane acoustic waves, seems an obvious one but does not appear to be readily available in the literature.

It is clear that if the supporting stresses were uniformly distributed over the plane their effect could be accounted for by a modified surface response equation. The previous conclusion that the radiation would be purely quadrupole would then be valid. It seems that it is essentially the inhomogeneous nature of the supports that induces the dipole component. If the plane surface were composed of several regions of locally homogeneous material, but material that differed from region to region, one could conclude from the foregoing analysis that within individual regions the effect of surface motion would be accounted for by the local reflexion coefficient and that the radiation would be quadrupole. However at the interfaces there would be discontinuities in the response equation that would account for dipole terms which must be more efficient radiators of sound. The situation is completely analogous to that treated by Maidanik (1962), who showed how most of the sound radiated from a large finite plate appeared to emanate from the periphery of the plate.

The total dipole strength is the net applied force and the most effective radiation results from the force being concentrated on to an area of typical dimension small in comparison with an acoustic wavelength. The concentrated point support is then an important example in establishing an upper limit on the strength of the radiation field induced by a known supporting force. That example is considered below.

2. Point-supported surface under a turbulent boundary layer

The simplest of the inhomogeneous support systems is that in which the stress distribution q is concentrated at one point. Let that point be the origin of co-ordinates and let the applied force have a value Q:

$$q(\mathbf{y},t) = Q(t)\,\delta(\mathbf{y}). \tag{2.1}$$

This force induces a dipole radiation according to equations (1.4) and (1.5), where the real and image fields become

$$S_+ = T_+ + \frac{1}{4\pi}\frac{\partial}{\partial x_n}\left[\frac{Q}{r}\right], \quad S_- = T_- - \frac{1}{4\pi}\frac{\partial}{\partial x_n}\left[\frac{Q}{r}\right]. \tag{2.2}$$

In the distant radiation field, the differentiation with respect to x_n applies only to the retarded time so that the dipole term may be rewritten as

$$\frac{1}{4\pi r}\frac{\sin\theta}{a_0}\frac{\partial Q}{\partial t}: \tag{2.3}$$

$\sin\theta$ is written for $-\partial r/\partial x_n$, θ being the radiation angle measured from the surface. The spectral decomposition is achieved through Fourier transformation. We shall denote transformed quantities by an asterisk, p^* being the component of p at frequency ω, T^*_+ the component of T, etc.

$$p(\mathbf{x},t) = \int p^*(\mathbf{x},\omega)\,e^{i\omega t}\,d\omega. \tag{2.4}$$

The particular case of equation (1.6) is then

$$p^*(\mathbf{x},\omega) = T^*_+ + RT^*_- + [1-R]\frac{i\omega\sin\theta}{4\pi a_0 r}Q^*. \tag{2.5}$$

It is easy to verify that the dipole term is precisely that worked out by Maidanik & Kerwin (1966) to be the radiation from a point-driven plate, but its interpretation as the sum of direct and reflected dipole fields seems to be new. A point of considerable interest is the question of how large the dipole field is in comparison with the sound radiated by the surrounding surface. To progress with that issue, the value of the externally applied force, Q, must first be found.

Suppose the support to have some impedance z_q, so that Q^* is related to the velocity at the support point v^*_{nq}, through the relation

$$Q^* = z_q v^*_{nq}. \tag{2.6}$$

The applied force has modified the velocity at its point of application, from a value v_{no}, which would have occurred in an unsupported structure, to its current value v_{nq}. The force is related to this velocity change through z_p, the point impedance of the structure, an impedance that includes any influence of fluid loading:

$$Q^* = -z_p(v^*_{nq} - v^*_{no}). \tag{2.7}$$

Combining this relation with equations (2.5) and (2.6) we obtain expressions for the applied force and radiated sound in terms of the velocity response of an unsupported structure:

$$Q^* = \frac{z_p}{1+(z_p/z_q)} v^*_{no}, \tag{2.8}$$

$$p^*(\mathbf{x},\omega) = T^*_+ + RT^*_- + \{1-R\}\frac{i\omega \sin\theta}{4\pi a_0 r}\frac{z_p z_q}{z_p+z_q} v^*_{no}. \tag{2.9}$$

The influence of the support can now be interpreted as a wave-scattering process. Sound radiation from an unsupported surface occurs only from those spectral components that match both frequency and wave number of the distant acoustic wave. But, in this instance, components of response velocity at all wave numbers contribute to the term v^*_{no}, so that the supported structure acts like a sounding board. That is, energy is converted from a reactive to a radiative régime by a wave-scattering process. This feature destroys any significant correlation between the dipole and quadrupole terms making the mean square radiation the sum of the individual mean square values. The power spectral density of the radiated pressure field $P^*(\mathbf{x},\omega)$ can then be expressed as the sum of that due to the combination of real and image quadrupoles T^*, and that due to the dipole which is proportional to the power spectral density of the response velocity in an unsupported panel, V^*_0:

$$P^*(\mathbf{x},\omega) = T^* + |1-R|^2 \frac{\omega^2 \sin^2\theta}{16\pi^2 a_0^2 r^2}\left|\frac{z_p z_q}{z_p+z_q}\right|^2 V^*_0. \tag{2.10}$$

V^*_0 is simply related to the three-dimensional Fourier spectrum of the pressure field acting on a rigid surface through the surface and wave impedances z and z_w, respectively. The rigid-surface pressure field at wave vector $\mathbf{k}$ and frequency ω, $p^*_{rs}(\mathbf{k},\omega)$, must balance both the structural response force, $zv^*_n(\mathbf{k},\omega)$, and the force induced by fluid motion, $z_w v^*_n(\mathbf{k},\omega)$. Therefore

$$p^*_{rs}(\mathbf{k},\omega) = (z+z_w)\, v^*_n(\mathbf{k},\omega), \tag{2.11}$$

where $v^*_n(\mathbf{k},\omega)$ is the three-dimensional Fourier transform of the surface velocity. The three-dimensional spectral functions, which we denote $P^*_{rs}(\mathbf{k},\omega)$ and $V^*_n(\mathbf{k},\omega)$, are formed from the product of this equation with its complex conjugate

$$P^*_{rs}(\mathbf{k},\omega) = |z+z_w|^2\, V^*_n(\mathbf{k},\omega). \tag{2.12}$$

The frequency spectrum of the surface response velocity is simply the integral of the three-dimensional spectrum over all wave-number space so that $V^*_0(\omega)$ is given by a straightforward integral,

$$V^*_0(\omega) = \int \frac{P^*_{rs}(\mathbf{k},\omega)}{|z+z_w|^2}\, d\mathbf{k}. \tag{2.13}$$

For a homogeneous structure the impedance function depends only on the magnitude of the wave vector so that a change of co-ordinates to a polar system is suggested. We let $\mathbf{k}$ be defined by (k,ϕ) and rewrite $d\mathbf{k}$ as $k\,dk\,d\phi$.

$$V^*_0(\omega) = \int_0^\infty \frac{k\,dk}{|z+z_w|^2}\int_0^{2\pi} P^*_{rs}(\mathbf{k}(k,\phi),\omega)\, d\phi. \tag{2.14}$$

In a highly resonant structure, the integral over wave number can be approximated in a way that considerably simplifies the analysis. The approximation rests on the assumption that both the pressure spectrum and the real part of the impedance $(z+z_w)$, (z_R+z_w) (z_w being necessarily real for radiating waves) remain fairly constant over the effective bandwidth of the 'resonance peak', a peak assumed to occur at wave number k_p. In that event, the integration over wave number is straightforward:

$$V_0^*(\omega) = \frac{2\pi}{\{z_R+z_w\}z_p}\int_0^{2\pi} P_{rs}^*(\mathbf{k}(k_p,\phi),\omega)\,d\phi. \tag{2.15}$$

Most turbulent flows of practical interest display convective features that are apparent in the spectrum function as a tendency for the energy to be concentrated at the eddy passage frequency. This property can be important in the response problem and is best dealt with by re-expressing the pressure spectrum in terms of that measured by an observer in uniform motion with the most coherent eddy structure. That spectrum we shall denote by $P_m^*(\mathbf{k},\omega)$.

$$P_{rs}^*(\mathbf{k},\omega) = P_m^*(\mathbf{k},\omega-\mathbf{k}\,.\,\mathbf{U}_c). \tag{2.16}$$

$\mathbf{U}_c$ is the convection velocity, which we normalize with respect to the free-surface wave speed, c_p. $|U_c| = c_p M$. Then, by setting the origin of ϕ coincident with the direction of convection and noting that $\omega = k_p c_p$, we can rewrite equation (2.15) in a form that displays the convective effects more clearly:

$$V_0^*(\omega) = \frac{2\pi}{\{z_R+z_w\}z_p}\int_0^{2\pi} P_m^*(\mathbf{k}(k_p,\phi),\omega\{1-M\cos\phi\})\,d\phi. \tag{2.17}$$

At low flow velocities, particularly in underwater applications, interest centres on situations where the number M is negligible. Then it is convenient to assume that space and time variables are separable in the moving reference frame so that

$$P_m^*(\mathbf{k}(k_p,\phi),\omega\{1-M\cos\phi\}) \simeq p_h^2 P_{\mathbf{k}}^*(\mathbf{k})\,P_\omega^*(\omega). \tag{2.18}$$

p_h is written for the R.M.S. pressure level active on a rigid boundary, $P_{\mathbf{k}}^*$ is the wave-number spectrum and P_ω^* is the moving axis frequency spectrum. Both these spectral functions are normalized to integrate to unity.

Before going on to evaluate the integral at low values of M, it is worth pointing out the equivalence that exists between this theory of vibration induced by convected pressure fields and that of aerodynamic sound generation by convected turbulence (Lighthill 1962; Ffowcs Williams 1963). In the aerodynamic case, M is the Mach number of eddy convection, and radiation frequencies differ from those of the source by the Doppler factor $(1-M\cos\phi)$. It is apparent from (2.17) that this feature is also a property of the vibration problem and that we might expect an analogue of the Mach wave radiation at values of M in excess of unity. This is evident from the alignment that occurs whenever $(1-M\cos\phi)$ approaches zero of the dominant spectral component in the surface pressure field with the response frequencies, the spectrum P_ω^* being chosen so that its maximum occurs at zero frequency. Then, by analogy with the aerodynamic problem, only the uniformly convected components induce response.

Consequently the response would be expected to be relatively intense for those waves radiating at the 'Mach angle' on account of the strong tendency to uniform convection of many pressure fields. The situation is illustrated quite effectively by assuming the moving-axis frequency spectrum to be a unit delta function. Then equation (2.17) can be evaluated to give the response velocity spectrum typical of structural excitation by high-speed flows where the convection velocity exceeds the phase speed of free waves:

$$V_0^*(\omega)_{(1-M\cos\phi)=0} = \frac{2\pi p_h^2 P_{\mathbf{k}}^*(k_p, \phi = \cos^{-1} M^{-1})}{\{z_R + z_w\} z_p \sqrt{(M^2-1)}}. \tag{2.19}$$

In underwater problems the low-speed situation is more relevant, where, from (2.17) and (2.18),

$$V_0^*(\omega) = \frac{2\pi p_h^2 P_\omega^*(\omega)}{\{z_R + z_w\} z_p} \int_0^{2\pi} P_{\mathbf{k}}^*(\mathbf{k}(k_p, \phi))\, d\phi. \tag{2.20}$$

The integral over ϕ is the correlation area $A(k_p)$ used by Ffowcs Williams & Lyon (1963), so that it is a previously estimated property of boundary-layer flows. The values for $A(k_p)$ are illustrated in figure 1,

$$2\pi \int_0^{2\pi} P_{\mathbf{k}}^*(\mathbf{k}(k_p, \phi))\, d\phi = A(k_p). \tag{2.21}$$

The response spectrum can now be given explicitly in terms of known features of the pressure field that acts on a rigid surface. The spectrum has the value derived for the flat plate by Ffowcs Williams & Lyon (1963) when z_w and z_p are appropriately chosen. The radiation from the simply supported surface can also be estimated by inserting the response spectrum in (2.10):

$$V_0^*(\omega) = [p_h^2 P_\omega^*(\omega) A(k_p)]/[\{z_R + z_w\} z_p], \tag{2.22}$$

$$P^*(\mathbf{x}, \omega) = T^* + |1-R|^2 \frac{\omega^2 \sin^2\theta}{16\pi^2 a_0^2 r^2} \frac{z_p}{\{z_R + z_w\}} \left|\frac{z_q}{z_q + z_p}\right|^2 p_h^2 P_\omega^*(\omega) A(k_p). \tag{2.23}$$

Although this result could be used to evaluate the sound radiated from various flows and support structures, the general form is not very revealing of the important role played by surface inhomogeneities. It is clear that, if the surface is supported on soft or resonant undamped mounts, so that z_q approaches zero, there will be no additional radiation from the support. It is also clear that the support plays a minor role in high-impedance structures where the reflexion coefficient approaches unity. However, many practical instances occur where the reflexion coefficient is close to zero, as in the case in sonar dome construction, where optimum sound transmission is essential. That situation is illustrated below by an example in which the support impedance z_q is infinite, the surface is assumed to be loss-free, and the flow is a fully developed turbulent boundary layer. The intensity of the radiation from the turbulence is identical with that from the image system which can be computed from a knowledge of the pressure field on the boundary surface. This has been done in an approximate form by Ffowcs Williams & Lyon (1963) with the result

$$T^* = \frac{p_h^2}{16\pi^2} \cos^2\theta \sin^2\theta \, 880\pi \frac{\delta_1^4}{\lambda^4} P_\omega^*(\omega) \frac{A}{r^2}. \tag{2.24}$$

The notation is that already defined with the addition that δ_1 is the boundary-layer displacement thickness, λ is the inverse acoustic wave number a_0/ω, and A is the area of the radiating surface. This estimate, when inserted in equation (2.23) yields an approximate expression for the total radiation from a turbulent boundary layer formed on a surface resting on a single rigid support:

$$P^*(\mathbf{x},\omega) = \frac{p_h^2}{16\pi^2}\frac{\delta_1^4}{r^2\lambda^2}\sin^2\theta P^*_\omega(\omega)\left\{880\pi\frac{A}{\lambda^2}\cos^2\theta+\frac{z_p}{\delta_1^2 z_w}\frac{A(k_p)}{\delta_1^2}\right\}. \tag{2.25}$$

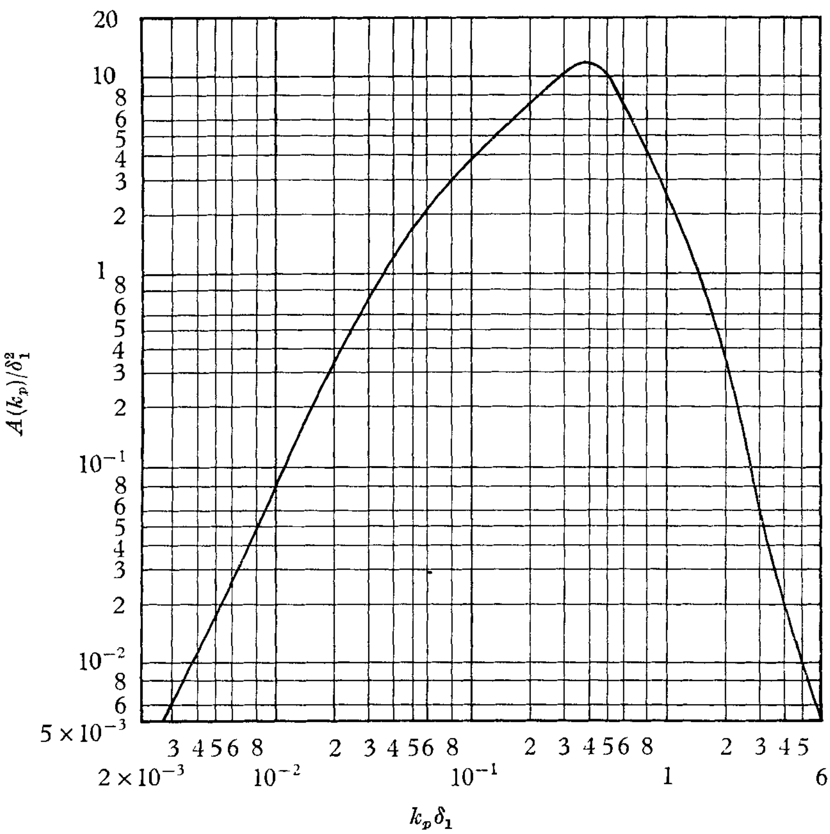

FIGURE 1. Ffowcs Williams & Lyon's (1963) estimate of the equivalent correlation area $A(k_p)$, defined by equation (2.21) of the text.

The leading term in the brackets is that due to the free surface and the other represents the support. When this expression is evaluated for a typical underwater situation one finds that the support induces an intensity at $\theta = 45°$ equivalent to that radiated by approximately 2 square metres of unsupported structure. This figure is worked out near the maximum value of $A(k_p)$, which occurs at a frequency of 3·5 kcycles for 0·25 in. steel plate, and the boundary-layer displacement thickness is taken as 0·1 in.

A more general result showing the area of free surface to which the radiation from the support is equivalent at a particular radiation angle is obtained by equating the two terms within the brackets of (2.25). The value of $A(k_p)/\delta_1^2$ has been taken as 10, that being an estimate of its upper limiting value as shown in figure 1. Then it is seen that at an angle θ the radiation from a rigid support on a surface with high transmission factor cannot exceed that from an unsupported area equal to

$$\frac{1}{88\pi\cos^2\theta}\frac{z_p}{z_w}\frac{\lambda^2}{\delta_1^2}. \tag{2.26}$$

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